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Feynman's variational method applied to the randomly forced Duffing equation

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Abstract. An approximation procedure, first introduced by Lücke for calculating the velocity correlation function in stationary homogeneous turbulence, is here applied to the Duffing equation driven by white noise. The approximation in question is based on the application of Feynman's variational principle to the functional integral representation of a generator for the correlation functions. The spectral function for a statistically stationary state is calculated for various values of the damping and non-linear coupling constants and compared with accurate results previously obtained by Bixon and Zwanzig and by Morton and Corrsin. The agreement is found to be poor except for the case of strong damping and weak non-linearity. Moreover, a simpler approximation obtained by statistical linearisation of the Duffing equation gives much better results.

1. Introduction

The Duffing equation describes a very simple non-linear system, namely a one-dimensional oscillator with linear and cubic terms in the restoring force and linear damping. For the driven oscillator the equation becomes, with a suitable choice of units,

$$\ddot{X} + \mu\dot{X} + X + \lambda X^3 = F(t).$$

If the force $F(t)$ is a stationary random function then a statistically stationary state is eventually attained in which viscous dissipation is balanced by the energy input from the driving force. The problem then arises of calculating the correlation functions such as $\langle X(t)X(t')\rangle$ for this state in terms of the known statistics of the force F. This model of a non-linear stochastic system has proved very useful in testing approximation schemes developed for more complicated physical systems. For example, Morton and Corrsin (1970) examined a large number of approximations ranging from simple linearisation to some rather elaborate methods based on truncated renormalised perturbation theory, while Bixon and Zwanzig (1971) have considered an approximation procedure based on the projection operator theory of Zwanzig and Mori. There is very good agreement between the best theoretical results in these two papers and we assume that these give a close approximation to the truth. This view is also supported by the analogue computations of Morton and Corrsin.

In the present paper we use the same model to test an approximation method introduced by Lücke (1978a) for the problem of stationary homogeneous turbulence. This is based on an application of Feynman's variational principle to a functional integral representation of a generator for the correlation functions. Calculations have been carried out for the turbulence problem (Lücke 1978a, b, Lücke and Zippelius 1978) with encouraging results. However, the trial function used in this work is of an unnecessarily restricted form, and the removal of the restrictions leads to a more complicated approximation (Phythian 1980). No calculations using this have yet been carried out. It seems appropriate therefore to test this general procedure on something simpler than turbulence, and the model described above is a suitable choice.

In § 2 the approximation is derived starting from the functional integral representation first obtained by Graham (1973). In § 3 calculations of the spectral function for the case of white noise forcing are presented in graphical form, for various values of the damping μ and the non-linear coupling constant λ, and compared with the results obtained by Bixon and Zwanzig. The agreement is found to be poor except for strong damping and weak non-linearity. Moreover, it is found that a simpler approximation, based on statistical linearisation of the Duffing equation, gives much more accurate results.

2. Derivation of the approximation

Following previous work we shall assume the random force $F(t)$ to be a Gaussian random function of zero mean and with correlation function

$$\langle F(t)F(t')\rangle = R(t-t').$$

The approximation will be derived for general R but the numerical results obtained will be for the particular case of white noise. First of all the basic equation is rewritten as a first-order equation

$$\dot{X}_\alpha(t) = F_\alpha(t) + \Lambda_\alpha(X(t)) \qquad \text{for } \alpha = 1, 2 \tag{1}$$

where

$$\begin{aligned} &F_1(t) = 0 && F_2(t) = F(t) \\ &X_1(t) = X(t) && X_2(t) = \dot{X}(t) \\ &\Lambda_1(X) = X_2 && \Lambda_2(X) = -X_1 - \mu X_2 - \lambda X_1^3. \end{aligned} \tag{2}$$

A slight generalisation of Graham's result has been derived (Phythian 1977) for (1) which shows that averages of functionals of X may be calculated by means of a probability density functional given by

$$P[x] = \mathcal{N}^{-1}\int D[\phi] \exp\left\{-\tfrac{1}{2}\int \mathrm{d}t \int \mathrm{d}t' \phi_\alpha(t) R_{\alpha\beta}(t-t')\phi_\beta(t') - \tfrac{1}{2}\int \mathrm{d}t\, \Lambda_{\alpha,\alpha}(x(t)) \right.$$
$$\left. -\mathrm{i}\int \mathrm{d}t\, \phi_\alpha(t)[\dot{x}_\alpha(t) - \Lambda_\alpha(x(t))]\right\} \tag{3}$$

where $\mathcal{N}$ is an infinite normalisation constant, $R_{\alpha\beta}(t-t')$ denotes the correlation function $\langle F_\alpha(t)F_\beta(t')\rangle$, and the functional integration denoted by $\int D[\phi]$ is to be carried out over the subsidiary functions $\phi_\alpha(t)$. Substituting from (2) into (3), performing the Gaussian integration over ϕ_2, and using the fact that the integration over ϕ_1 gives the delta functional $\delta[\dot{x}_1 - x_2]$ we obtain

$$P[x] \propto \exp\left(-\frac{1}{2}\int \mathrm{d}t \int \mathrm{d}t'\, y(t)Q(t-t')y(t')\right) \tag{4}$$

where

$$y(t) = \ddot{x} + \mu\dot{x} + x + \lambda x^3 \tag{5}$$

and Q denotes the inverse of the correlation function R, i.e.

$$\int \mathrm{d}t'\, Q(t-t')R(t'-t'') = \delta(t-t'').$$

Averages are then given by functional integrals of the form

$$\langle F[x]\rangle = \int D[x]F[x]P[x]$$

where the integration is to be carried out over all functions $x(t)$, the initial conditions having been banished to the infinitely remote past so that the statistically stationary state has been attained.

Correlation functions may be conveniently expressed in terms of the generating functional

$$Z[h] = \int D[x]\,\mathrm{e}^{-S} \tag{6}$$

where S is given by the expression

$$\int \mathrm{d}t\, h(t)x(t) + \frac{1}{2}\int \mathrm{d}t \int \mathrm{d}t'\, y(t)Q(t-t')y(t'). \tag{7}$$

For example, we have

$$\langle X(t)X(t')\rangle = \left(\frac{1}{Z}\frac{\delta^2 Z}{\delta h(t)\delta h(t')}\right)_{h=0}.$$

The mean value $\langle X(t)\rangle$ is seen to be zero since $P[x]$ is an even functional of x. The basic idea of Lücke's theory is to use the Feynman variational principle to obtain an approximation for Z from (6). We have the inequality

$$\int D[x]\,\mathrm{e}^{-S} \geqslant \left(\int D[x]\,\mathrm{e}^{-S_0}\right)\mathrm{e}^{-\langle S-S_0\rangle_0} \tag{8}$$

where $\langle \ldots \rangle_0$ denotes an average performed with the probability density

$$\mathrm{e}^{-S_0}\Big/\int D[x]\,\mathrm{e}^{-S_0}$$

and S_0 is a suitable trial functional. In order that the integrals be calculable it is necessary to take S_0 as a sum of linear and quadratic terms in x. However, by optimising the inequality with respect to the coefficient functions in S_0 one hopes to obtain a useful approximation for Z. We therefore take

$$S_0 = \int \mathrm{d}t\, \eta(t)x(t) + \frac{1}{2}\int \mathrm{d}t \int \mathrm{d}t'\, A(t,t')x(t)x(t') \tag{9}$$

where η is arbitrary and A is a positive definite kernel, so that

$$\int D[x]\,\mathrm{e}^{-S_0} = \text{constant} \times \sqrt{\det G} \exp\left[\tfrac{1}{2}\int\int \eta(t)G(t,t')\eta(t')\right]$$

where G is the kernel inverse to A. This may be rewritten

$$\text{constant} \times \exp\left[\tfrac{1}{2}\operatorname{Tr}\lg G + \tfrac{1}{2}\int\int \eta(t)G(t,t')\eta(t')\right]. \tag{10}$$

To proceed further it is convenient to introduce some diagram representations. Let us first rewrite (5) in the form

$$y(t) = \int q(t-t_1)x(t_1) + \int M(tt_1t_2t_3)x(t_1)x(t_2)x(t_3)$$

where integrations are to be carried out over repeated t's, and where

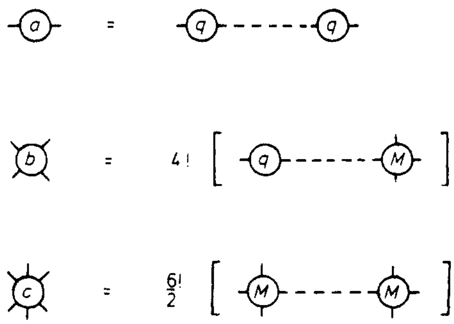
$$q(t-t_1) = \left(\frac{\partial^2}{\partial t^2} + \mu\frac{\partial}{\partial t} + 1\right)\delta(t-t_1)$$

$$M(tt_1t_2t_3) = \lambda\delta(t-t_1)\delta(t-t_2)\delta(t-t_3).$$

The expression for S may now be written

$$S = \int h(t_1)x(t_1) + \frac{1}{2!}\int a(t_1t_2)x(t_1)x(t_2) + \frac{1}{4!}\int b(t_1t_2t_3t_4)x(t_1)\ldots x(t_4)$$
$$+\frac{1}{6!}\int c(t_1\ldots t_6)x(t_1)\ldots x(t_6) \tag{11}$$

where a, b, c are symmetric functions given by the following diagrams:



(12)

Here the broken line denotes Q and the square brackets indicate that the expression be symmetrised. For example, the equation for b given in full is

$$b(t_1t_2t_3t_4) = 6\lambda\int [q(t-t_1)Q(t-t')\delta(t_2-t')\delta(t_3-t')\delta(t_4-t')$$
$$+q(t-t_2)Q(t-t')\delta(t_3-t')\delta(t_4-t')\delta(t_1-t')$$
$$+q(t-t_3)Q(t-t')\delta(t_4-t')\delta(t_1-t')\delta(t_2-t')$$
$$+q(t-t_4)Q(t-t')\delta(t_1-t')\delta(t_2-t')\delta(t_3-t')].$$

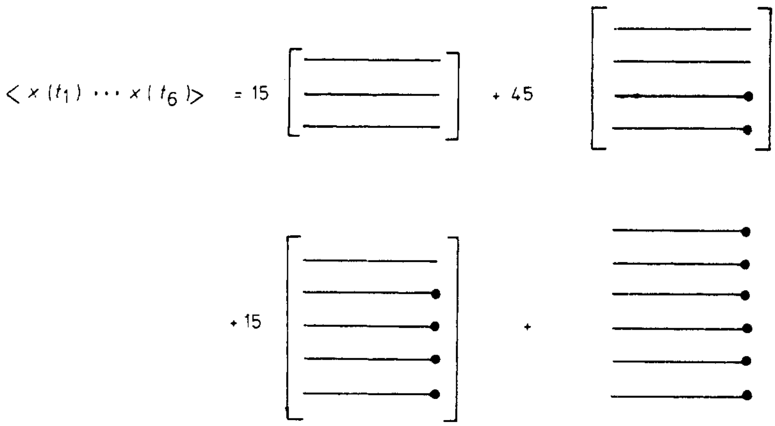
The quantity $\langle S-S_0\rangle_0$ clearly involves averages of products of x's with as many as six factors. Denoting $G(t, t')$ by t——t' and $\eta(t)$ by t-● it is easily verified that

$$\langle x(t_1)\rangle_0 = -t_1\text{———}\bullet = -\int G(t_1, t)\eta(t)$$

$$\langle x(t_1)x(t_2)\rangle_0 = t_1\text{———}t_2 + \begin{matrix} t_1\text{———}\bullet \\ t_2\text{———}\bullet \end{matrix} \tag{13}$$

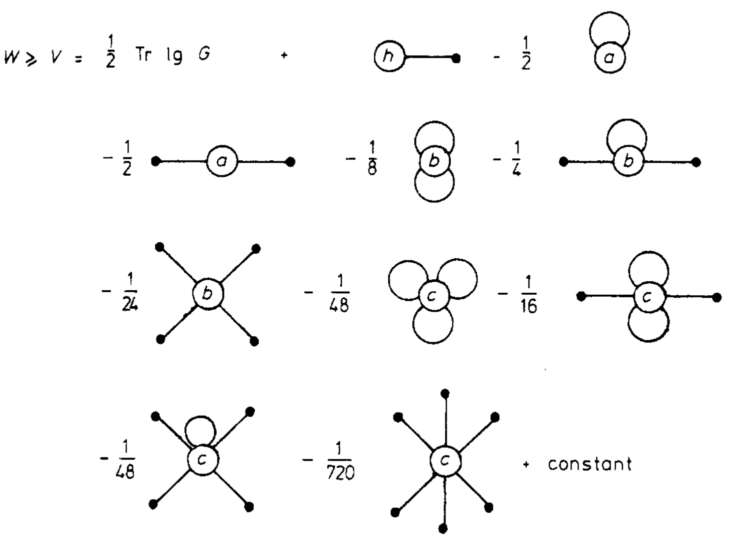
$$= G(t_1, t_2) + \int G(t_1, t)G(t_2, t'), \eta(t')\eta(t)$$

$$\vdots$$



Again the brackets indicate a symmetrisation in the t's of the line endings.

After some straightforward but tedious analysis using (10), (11) and (13) we emerge with the following lower bound on $W = \lg Z$

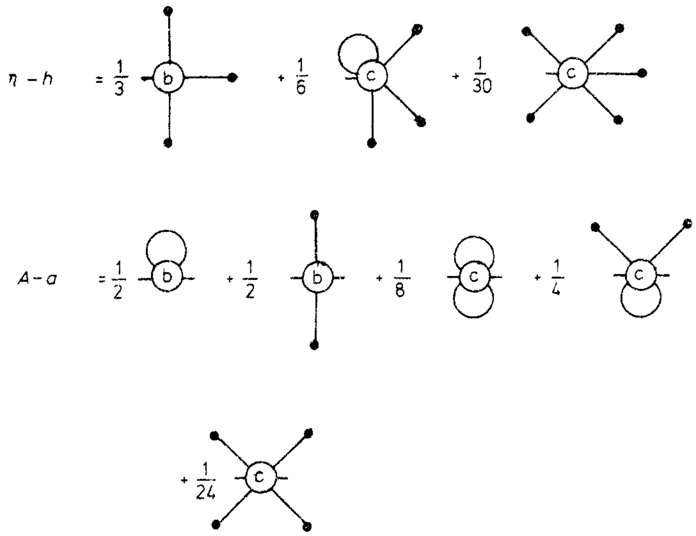


(14)

A stationary value of V with respect to variations in η and A is given by the conditions

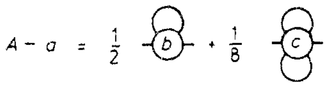
$$\delta V/\delta\eta = 0 \qquad \delta V/\delta A = 0$$

which lead to the equations



(15)

For $h=0$ it is seen that a particular solution of (15) is given by $\eta=0$ with A satisfying the equation



(16)

We shall assume that the maximum of V is attained, at least for sufficiently small h, for the solution of (15) which reduces as $h \to 0$ to this particular solution. In principle it should be possible to check this by examining the second-order variation of V but the analysis is complicated and will not be attempted here. Our approximation for W is then given by (14) with η and A (and hence G) expressed in terms of h by means of (15).

Denoting this approximation by $\bar{W}$ we can now calculate approximate correlation functions by differentiating $\bar{W}$ with respect to h. Using a slightly abbreviated notation we have

$$\frac{\delta \bar{W}}{\delta h}=\frac{\delta V}{\delta h}+\frac{\delta V}{\delta G}\frac{\delta G}{\delta h}+\frac{\delta V}{\delta \eta}\frac{\delta \eta}{\delta h}$$

which, because of the stationary point conditions, is just $\delta V/\delta h$. Using (14) this is seen to be $G\eta$. The approximate mean value $[\delta \bar{W}/\delta h]_{h=0}$ is therefore $[G\eta]_{h=0}$ which is zero, as expected, since $\eta=0$ for $h=0$. We now calculate the second derivative:

$$\frac{\delta^2 \bar{W}}{\delta h^2}=\frac{\delta^2 V}{\delta h^2}+\frac{\delta G}{\delta h}\frac{\delta^2 V}{\delta G \delta h}+\frac{\delta \eta}{\delta h}\frac{\delta^2 V}{\delta \eta \delta h}$$

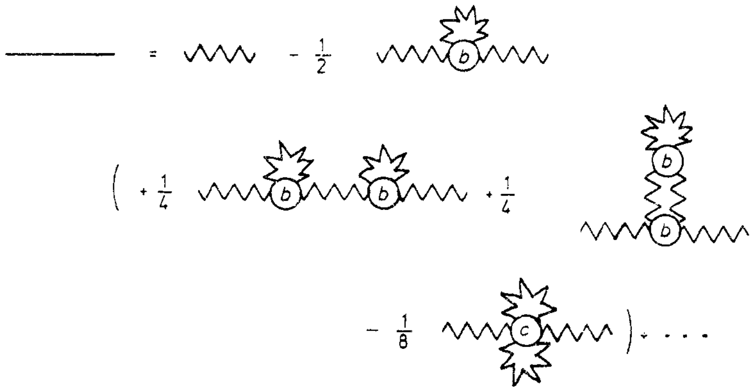
$$=\eta\frac{\delta G}{\delta h}+G\frac{\delta \eta}{\delta h}.$$

Setting $h=0$ and using (15) gives

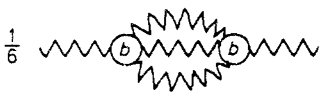
$$\left[\frac{\delta^2 \bar{W}}{\delta h^2}\right]_{h=0}=[G]_{h=0}.$$

Thus our approximate correlation function is just G determined by (16). The approximation has a simpler structure than the one derived in the same way for the turbulence problem (Phythian 1980). There it was found necessary to introduce a type of vertex function and to solve two coupled equations for this quantity and the correlation function. The difference seems to arise from the fact that the non-linearity in the present case is cubic, rather than quadratic as in the Navier Stokes equation, so that the probability density functional is even in x.

Before deriving the final form of the equation it is instructive to write down the perturbative solution for G. Denoting the correlation function for the case of zero λ, which is seen to be a^{-1}, by  we derive from (16) the series



The exact series obtained by expanding the functional integrals as power series in λ, or directly from the equation of motion, agrees with that given above to order λ^2 except for the appearance of an extra term



in second order.

The final form of the approximation is obtained by making use of the expressions for a, b, c given in (12). In terms of Fourier transforms defined, for example, by

$$G(t)=\frac{1}{2\pi}\int \mathrm{d}\omega\, \tilde{G}(\omega)\, \mathrm{e}^{\mathrm{i}\omega t}$$

($\tilde{G}(\omega)$ is therefore our approximation for the spectral function) we get

$$\begin{aligned}\tilde{A}(\omega)-\tilde{a}(\omega)=&\frac{3\lambda}{\pi}\int \tilde{G}(\omega')[\tilde{Q}(\omega)(1-\omega^2)+\tilde{Q}(\omega')(1-\omega'^2)]\\&+\frac{9\lambda^2}{4\pi^2}\int\int \tilde{G}(\omega')\tilde{G}(\omega'')[\tilde{Q}(\omega)+\tilde{Q}(\omega')+\tilde{Q}(\omega'')+2\tilde{Q}(\omega+\omega'+\omega'')]\end{aligned} \tag{17}$$

with

$$\tilde{G}(\omega)=1/\tilde{A}(\omega)$$

and

$$\tilde{a}(\omega)=[(1-\omega^2)^2+\mu^2\omega^2]\tilde{Q}(\omega).$$

3. Numerical results

We now assume the random force $F(t)$ to be a white noise function and, without loss of generality, we take

$$R(t)=\delta(t)$$

so that

$$\tilde{Q}(\omega)=1.$$

Putting

$$\int \tilde{G}(\omega)=\beta$$

$$\int (1-\omega^2)\tilde{G}(\omega)=\gamma$$

we have

$$\tilde{G}(\omega)=\left((1-\omega^2)^2+\mu^2\omega^2+\frac{3\lambda}{\pi}(\beta+\gamma-\omega^2\beta)+\frac{45}{4\pi^2}\lambda^2\beta^2\right)^{-1}.$$

A simple algebraic transformation now gives

$$\tilde{G}(\omega)=\frac{1}{\omega^4-2\omega^2(\xi-2\psi^2)+\xi^2} \tag{18}$$

where ξ and ψ are both positive and satisfy

$$\begin{aligned} \xi-2\psi^2&=1-\frac{\mu^2}{2}+\frac{3\lambda}{4\xi\psi} \\ \xi^2&=1+\frac{3\lambda}{2\psi}\left(\frac{2}{\xi}-1\right)+\frac{45\lambda^2}{16\xi^2\psi^2}. \end{aligned} \tag{19}$$

We have solved these equations for the following pairs of values (μ, λ): (2, 0·25), (2, 1), (2, 2), (2, 3), (0·5, 0·1), (0·5, 0·25), and the corresponding spectral functions $\tilde{G}(\omega)$ are plotted in figures 1–6 together with the best theoretical results of Bixon and Zwanzig. It will be seen that the approximate spectral function is positive, as required by realisability, and displays the correct qualitative shape in all cases but the numerical agreement is rather poor except in the case of large damping and weak non-linearity.

The graphs also show for comparison the approximation obtained by statistical linearisation. This follows from the replacement of the Duffing equation by the linearised one:

$$\ddot{X}+\mu\dot{X}+X+3\lambda\langle X^2\rangle X=F(t)$$

and is described by Morton and Corrsin (1970) and, more recently and extensively, by Budgor *et al* (1976). This approximation also gives a spectral function of the form (18) but with ξ and ψ now given by

$$\psi=\frac{\mu}{2} \qquad \xi=1+\frac{3\lambda}{4\xi\psi}.$$

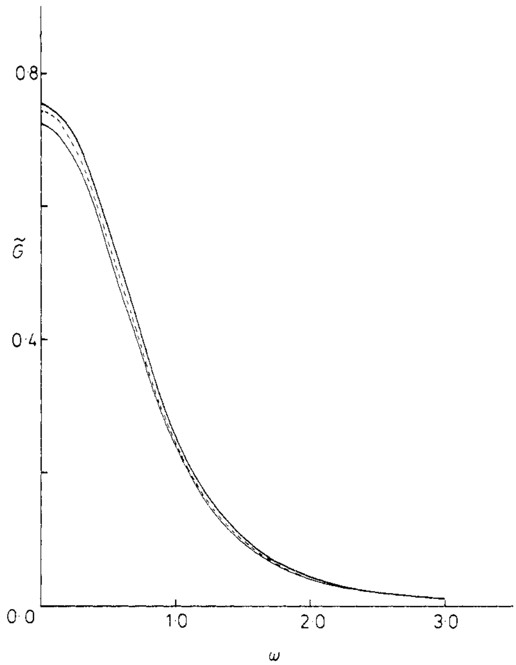


Figure 1. The spectral function $\tilde{G}(\omega)$ plotted as a function of ω for the case $\mu = 2$, $\lambda = 0{\cdot}25$. The upper continuous curve shows the 'exact' results of Bixon and Zwanzig and the lower continuous curve the present approximation. The broken curve shows the results obtained from an alternative approximation based on statistical linearisation.

Note that this choice of ξ and ψ satifies the first of equations (19) but not the second. It will be apparent from the graphs that this approximation, while simpler than that derived from the variational approach, is much superior.

4. Conclusion

The Feynman variational principle has been used above, in the manner suggested by Lücke in his treatment of the turbulence problem, to derive an approximate spectral function for the Duffing equation driven by white noise. Although the approximation is of a simple form and gives a positive spectral function its numerical accuracy is poor. Moreover, it was seen that an even simpler approximation obtained by a straightforward statistical linearisation of the Duffing equation is far superior. At first sight this seems rather strange since the variational method, with a general quadratic trial functional, might be expected to yield the best approximation obtainable by linearisation. The basic difficulty seems to lie in the fact that the variational method is used to approximate a mathematical construct, the generating functional, rather than the quantities of interest (the correlation functions). Since the correlation functions are obtained by differentiation of the generating functional it is clearly possible to have a good approximation for the latter which gives a poor approximation for the former. It is of interest to note that a closer inspection of the exact and approximate functionals W

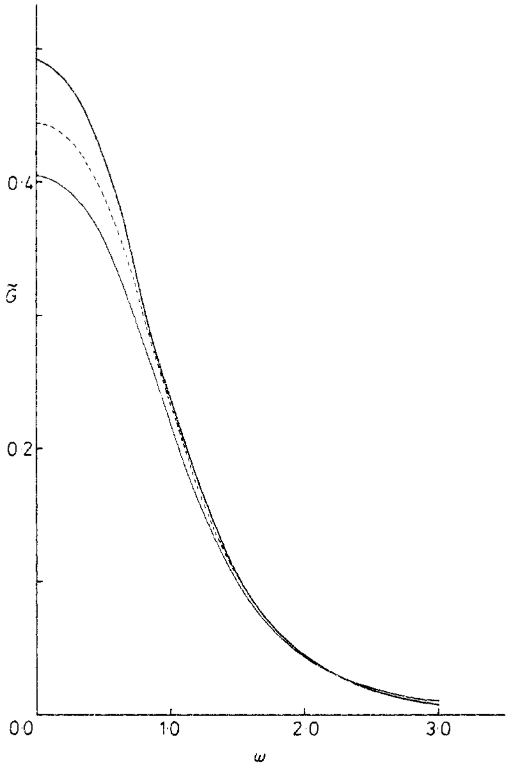


Figure 2. As for figure 1 but with $\mu = 2$, $\lambda = 1$.

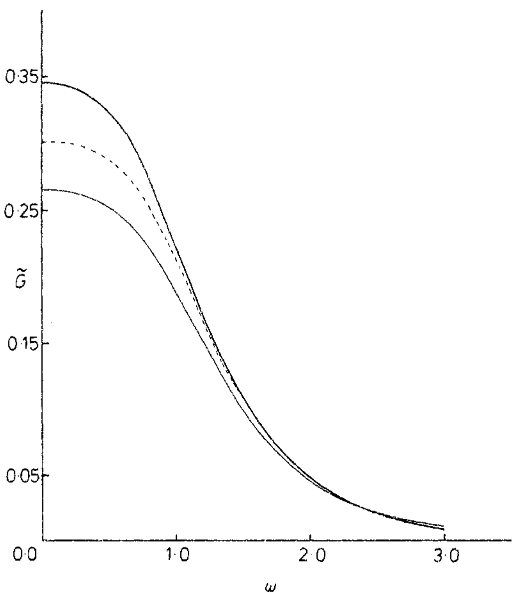


Figure 3. As for figure 1 but with $\mu = 2$, $\lambda = 2$.

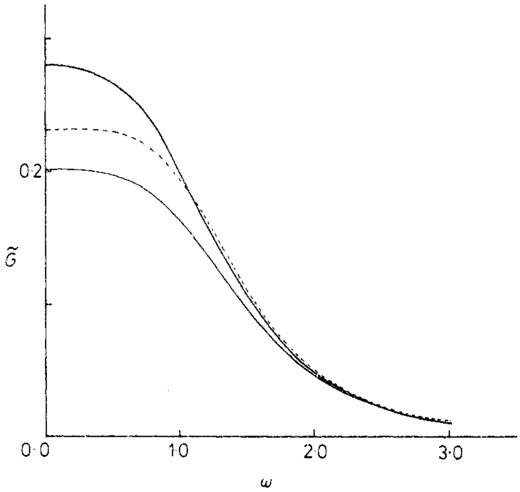


Figure 4. As for figure 1 but with $\mu = 2$, $\lambda = 3$.

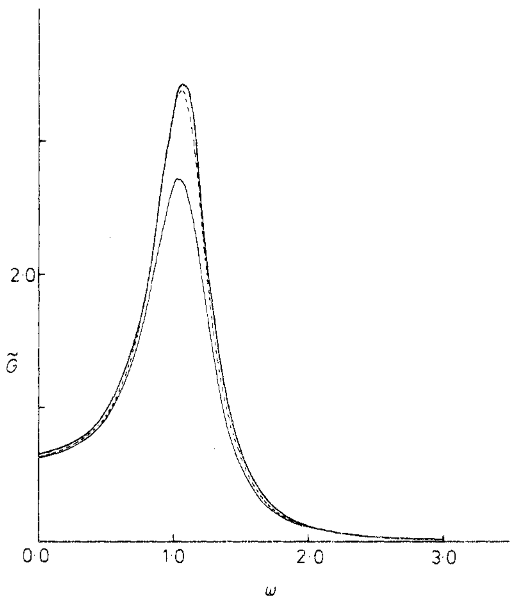


Figure 5. As for figure 1 but with $\mu = 0{\cdot}5$, $\lambda = 0{\cdot}1$.

and $\bar{W}$, using perturbation theory, suggests that they differ by an infinite quantity so the method may not even give a good approximation for the generating functional.

There are no obvious reasons for believing that the method would be more successful for other non-linear stochastic problems such as turbulence. Moreover, as was mentioned earlier, the approximation loses its simplicity in the case of turbulence unless arbitrary restrictions are imposed on the trial functional as in the theory of Lücke. We conclude that the method is probably unreliable as a quantitative theory of turbulence and should perhaps best be regarded as providing yet another realisable model which may have qualitative features in common with real turbulence.

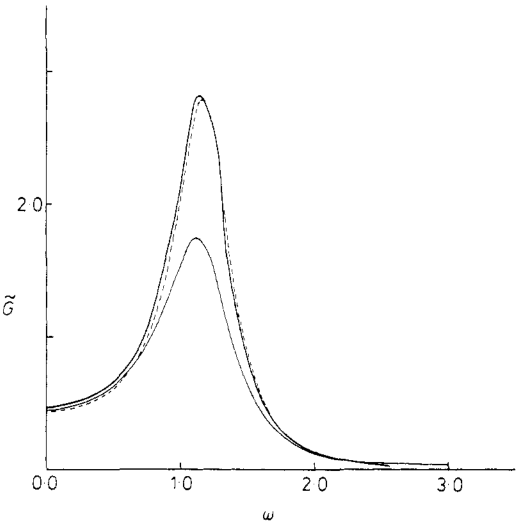


Figure 6. As for figure 1 but with $\mu = 0{\cdot}5$, $\lambda = 0{\cdot}25$.

Acknowledgment

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